

Nonlinear constraints in nonholonomic mechanics

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Abstract

Considering nonlinear nonholonomic constraints, a simple form of equations of regular dynamics is given, based on some Chetaev-like conditions. In the particular cases of linear and affine constraints, one obtains the classical equations in the forms given, for example, by Bloch and Marsden. The case of time-dependent constraints is also considered. Examples of linear constraints, time independent and time dependent nonlinear constraints are considered, as well as their dynamics given by suitable Lagrangians. All the examples are based on classical ones, such as those given by Appell's machine.

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1 Introduction

The geometrization of nonholonomic systems is a historical outstanding problem in mechanics and geometry (see, for example [16]). In general, the most used and studied constraints in nonholonomic mechanics are linear and affine ones (see, for example, [2, 4, 5, 6, 7, 9, 10, 18, 22]). But nonlinear constraints are also involved in nonholonomic mechanics, beside the linear and affine ones (see, for example, [3, 8, 11, 12, 13, 14, 15, 17, 19, 21, 20]).

The possibility to involve a nonlinear constraint and a Lagrangian that rules the dynamics is usually associated with Chetaev or generalized Chetaev principles. A criticism of Chetaev principle is performed, for example, in [21, 20], where some situations (as Appell machine) are presented as examples when Chetaev principle fails to a real situation. Other authors use Chetaev principle in some special conditions, as for example in [19], as a generalized Chetaev principle, when the constraint is homogeneous in the relative velocities and the constraints are time dependent. Our goal in the paper is not to study the workability of Chetaev or generalized Chetaev principle, but the possibility to put in a unitary form the dynamics equations coming from linear, affine and regular nonlinear constraints (Theorem 4.1).

The Chetaev principle, generally accepted in nonlinear constraint case, comes from the following principle: taking the variation before imposing the constraints, that is, not imposing the constraints on the family of curves defining the variation. In this case, one follows similar arguments as in the linear and affine constraints in [4, 5] and we give a new form expressed in Theorem 4.1. Adapting these results in the case of time dependent nonlinear constraints, we obtain a similar general result that applies in the cases of generalized Chetaev case [19, Section 2] or the example in [19, Section 3].

Some short preliminaries on foliations are given in the second section. Nonlinear constraints, including linear and affine ones, are considered for Lagrangians in the next section using foliations, but following the classical bundle setting as in [4] for linear and affine constraints. The implicit

forms of constraints and a link with the Lagrange multipliers form of Euler-Lagrange equation are also considered. For a nonlinear constraint C , a C -semispray S and an S -curvature R of C are defined using Proposition 3.3 and Proposition 3.4 respectively. Notice that in the cases of linear and affine constraints, the curvature need no semispray to be defined. A short form of nonholonomic Lagrangian dynamics, subject of linear and affine constraints, are presented in the third section, following [4] and it is extended in the case of dynamics generated by C -regular Lagrangians, having nonlinear constraint systems. The main result is Theorem 4.1, where a synthetic form of linear and regular-nonlinear cases is given. This result can be adapted to other situations; for example, in the case of time dependent constraints, but a time independent lagrangian (as studied in [19], see Example 3. in the last section). In order to illustrate the constructions performed in the paper, the cases of five well-known examples are discussed in the section five.

2 Preliminaries on foliations

Let us consider M an $(n + m)$ -dimensional manifold which will be assumed to be connected and orientable.

A codimension n foliation \mathcal{F} on M is defined by a foliated cocycle $\{U_i, \varphi_i, f_{i,j}\}$ such that:

- (i) $\{U_i\}$, $i \in I$ is an open covering of M ;
- (ii) For every $i \in I$, $\varphi_i : U_i \rightarrow T$ are submersions, where T is an n -dimensional manifold, called transversal manifold;
- (iii) The maps $f_{i,j} : \varphi_i(U_i \cap U_j) \rightarrow \varphi_j(U_i \cap U_j)$ satisfy

$$\varphi_j = f_{i,j} \circ \varphi_i \tag{2.1}$$

for every $(i, j) \in I \times I$ such that $U_i \cap U_j \neq \emptyset$.

Every fibre of φ_i is called a *plaque* of the foliation. Condition (2.1) says that, on the intersection $U_i \cap U_j$ the plaques defined respectively by φ_i and φ_j coincides. The manifold M is decomposed into a family of disjoint immersed connected submanifolds of dimension m ; each of these submanifolds is called a *leaf* of \mathcal{F} . If $U \subset M$ is an open subset, that a foliation \mathcal{F} on M induces a foliation \mathcal{F}_U on U , called an *induced foliation*.

By $T\mathcal{F}$ we denote the tangent bundle to \mathcal{F} and $\Gamma(\mathcal{F})$ is the space of its global sections i.e. vector fields tangent to \mathcal{F} .

A system of local coordinates adapted to the foliation \mathcal{F} means coordinates $(x^u, x^{\bar{u}})$, $u = 1, \dots, m$, $\bar{u} = 1, \dots, n$ on an open subset U on which the foliation is trivial and defined by the equations $dx^{\bar{u}} = 0$, $\bar{u} = 1, \dots, n$.

A particular example of a foliation is a *locally trivial fibration*. There are elementary examples of foliations that are not locally trivial fibrations and the spaces of leaves are not Hausdorff separated. For example, considering the natural projection $\pi_1 : \mathbb{R}^2 \rightarrow \mathbb{R}$, $(x, y) \xrightarrow{\pi_1} x$ one obtain a foliation \mathcal{F} on \mathbb{R}^2 ; but on $U = \mathbb{R}^2 \setminus \{(x, 0) | x \geq 0\} \subset \mathbb{R}^2$ the induced foliation \mathcal{F}_U is not a locally trivial fibration and the space of leaves is not Hausdorff separated (even if the leaves are fibers of a surjective submersion). According to the above conventions, the coordinates are denoted by $x = x^1$ and $y = x^{\bar{1}}$.

3 Linear, affine and nonlinear constraints and Lagrangians

A *linear constraint* system of a foliation \mathcal{F} is a left splitting of the inclusion $T\mathcal{F} \xrightarrow{I_0} TM$. Since there is a short exact sequence of vector bundle morphisms

$$0 \rightarrow T\mathcal{F} \xrightarrow{I_0} TM \xrightarrow{\Pi_0} N\mathcal{F} \rightarrow 0, \quad (3.2)$$

it follows that the existence a left splitting C of I_0 is equivalent with the existence a right splitting D of the projection Π_0 , thus a inclusion of $N\mathcal{F}$ in TM , via the injective morphism D , that gives a decomposition

$$TM = T\mathcal{F} \oplus N\mathcal{F},$$

where $N\mathcal{F} = D(N\mathcal{F})$. The *curvature* of C is the bilinear map $B : \Gamma(N\mathcal{F}) \times \Gamma(N\mathcal{F}) \rightarrow \Gamma(T\mathcal{F})$ given by

$$B(X, Y) = C([D(X), D(Y)]). \quad (3.3)$$

One say that a section $\bar{X} \in \Gamma(N\mathcal{F})$ is *transverse field* if for every vector fields $X, Y \in \mathcal{X}(M)$ such that $\bar{X} = \Pi_0(X)$ and $Y \in \Gamma(T\mathcal{F})$, then $[X, Y] \in \Gamma(T\mathcal{F})$; we say that $D(\bar{X}) \in \Gamma(N\mathcal{F})$ is the *horizontal lift* of \bar{X} . Thus if $\bar{X}, \bar{Y} \in \Gamma(N\mathcal{F})$ are transverse, the curvature has the form

$$B(\bar{X}, \bar{Y}) = C([\bar{X}^h, \bar{Y}^h]). \quad (3.4)$$

Using local coordinates, a linear constraint C has the local form

$$(x^u, x^{\bar{u}}, X^u, X^{\bar{u}}) \xrightarrow{C} (x^u, x^{\bar{u}}, X^u + C_{\bar{u}}^u(x^u, x^{\bar{u}})X^{\bar{u}}) \quad (3.5)$$

and the corresponding D is

$$(x^u, x^{\bar{u}}, X^{\bar{u}}) \xrightarrow{D} (x^u, x^{\bar{u}}, -C_{\bar{u}}^u(x^u, x^{\bar{u}})X^{\bar{u}}, X^{\bar{u}}) \quad (3.6)$$

The curvature B of C has the local form

$$\begin{aligned} B_{\bar{u}\bar{v}}^u \frac{\partial}{\partial x^{\bar{u}}} &= B\left(\frac{\delta}{\delta x^{\bar{u}}}, \frac{\delta}{\delta x^{\bar{v}}}\right) = \left[\frac{\delta}{\delta x^{\bar{u}}}, \frac{\delta}{\delta x^{\bar{v}}}\right] = \\ &= \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}} - \frac{\partial C_{\bar{v}}^u}{\partial x^{\bar{u}}} + C_{\bar{v}}^v \frac{\partial C_{\bar{u}}^u}{\partial x^v} - C_{\bar{u}}^v \frac{\partial C_{\bar{v}}^u}{\partial x^v}. \end{aligned} \quad (3.7)$$

where

$$\frac{\delta}{\delta x^{\bar{u}}} = \left(\frac{\partial}{\partial x^{\bar{u}}}\right)^h = \frac{\partial}{\partial x^{\bar{u}}} - C_{\bar{u}}^u(x^u, x^{\bar{u}}) \frac{\partial}{\partial x^u}.$$

As an example, we consider the linear Appell constraints (see, for example, [21]). The manifold is $M = \mathbb{R}^3 \times T^2$ and the foliation is the simple foliation defined by the fibers of the canonical projection $\mathbb{R}^3 \times T^2 \rightarrow T^2$. Consider the coordinates (x^1, x^2, x^3) on \mathbb{R}^3 and $(x^{\bar{1}}, x^{\bar{2}})$ on T^2 . The linear Appell constraints are given by the formulas

$$C^1 = Ry^{\bar{1}} \cos x^2, C^2 = Ry^{\bar{1}} \sin x^2, C^3 = ry^{\bar{1}}. \quad (3.8)$$

Using formulas (3.7), its curvature B has the coefficients

$$B_{\bar{1}\bar{2}}^1 = -R \sin x^2, B_{\bar{1}\bar{2}}^2 = R \cos x^2, B_{\bar{1}\bar{2}}^3 = 0. \quad (3.9)$$

An *affine constraint* system of a foliation \mathcal{F} is a fibered map $D' : N\mathcal{F} \rightarrow TM$ affine on fibers. One can decompose D' as

$$D'(\bar{X}) = D(\bar{X}) - b,$$

where D comes from a linear constraint $C : TM \rightarrow T\mathcal{F}$ and $b \in \Gamma(T\mathcal{F})$ is a tangent vector field to \mathcal{F} . We can define also a map $C' : TM \rightarrow T\mathcal{F}$, by $C'(X) = C(X) + b$. In the affine case, giving C and b is equivalent giving D and b , as easily can be seen.

In the similar way one can extend the definition of an adapted Lagrangian L , asking that L has the form

$$L(X) = L_0(C'(X)) + \bar{L}(\Pi_0(X)), X \in \mathcal{X}(\widetilde{TM}), \quad (3.10)$$

where C' is an affine constraint and $\widetilde{TM} = TM - \{\text{zero section}\}$.

According to [4, Ch. 5], a covariant derivative of b , along a horizontal vector field $\bar{X} \in \Gamma(D(N\mathcal{F}))$, can be considered as a vector field $\nabla_{\bar{X}}b \in \mathcal{X}(M)$ that projects by Π_0 on \bar{X} . Using local coordinates, if a linear constraint D has the local form (3.6), then C (corresponding to D) and D' have the forms (3.5) and

$$(x^u, x^{\bar{u}}, X^{\bar{u}}) \xrightarrow{D} (x^u, x^{\bar{u}}, b^u(x^u, x^{\bar{u}}) - C_{\bar{u}}^u(x^u, x^{\bar{u}})X^{\bar{u}}) \quad (3.11)$$

respectively. If

$$\bar{X} = \bar{X}^u \left(\frac{\partial}{\partial x^u} - C_{\bar{u}}^u \frac{\partial}{\partial x^{\bar{u}}} \right),$$

then

$$(x^u, x^{\bar{u}}) \xrightarrow{\nabla_{\bar{X}}b} (x^u, x^{\bar{u}}, \bar{X}^{\bar{v}} \gamma_{\bar{v}}^u, \bar{X}^{\bar{u}}),$$

where

$$\gamma_{\bar{u}}^u = \frac{\partial b^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial b^u}{\partial x^v} + b^v \frac{\partial C_{\bar{u}}^u}{\partial x^v}.$$

The curvature of D and the covariant derivative ∇ play an important role to write down in the next section the nonholonomic equations of motion. (See [4, Sect. 5.2] for more details.)

We deal now with nonlinear constraints.

Let us consider the endomorphism $\tilde{J} \in \text{End}(TN\mathcal{F})$, induced by the projection of the canonical almost tangent structure $J \in \text{End}(TTM)$. Using local coordinates, it is given by :

$$\frac{\partial}{\partial x^u} \xrightarrow{\tilde{J}} 0, \frac{\partial}{\partial x^{\bar{u}}} \xrightarrow{\tilde{J}} \frac{\partial}{\partial y^{\bar{u}}}, \frac{\partial}{\partial y^{\bar{u}}} \xrightarrow{\tilde{J}} 0. \quad (3.12)$$

Let $VN\mathcal{F} = \text{span}\{\frac{\partial}{\partial y^{\bar{u}}}\}$ be the vertical vector bundle of $N\mathcal{F}$. Denote by $\Gamma_0 \in \Gamma(VN\mathcal{F})$ the transverse Liouville vector field.

We say that a map $C : N\mathcal{F} \rightarrow TM$, viewed also as a section $C \in \Gamma(\pi_{N\mathcal{F}}^* TM)$, is a nonlinear constraint if $\tilde{J}(C) = \Gamma_0$. Using local coordinates,

$$\Gamma_0 = y^{\bar{u}} \frac{\partial}{\partial y^{\bar{u}}}, (x^u, x^{\bar{u}}, y^{\bar{u}}) \xrightarrow{C} (C^u(x^v, x^{\bar{v}}, y^{\bar{v}}), y^{\bar{u}}), C = C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}. \quad (3.13)$$

Proposition 3.1. *A nonlinear constraint give rise to a left splitting C'' or, equivalently, a right splitting D'' of the exact sequence of vector bundle morphisms*

$$0 \rightarrow \pi_{N\mathcal{F}}^* T\mathcal{F} \xrightarrow{I_0''} \pi_{N\mathcal{F}}^* TM \xrightarrow{\Pi_0''} \pi_{N\mathcal{F}}^* N\mathcal{F} \rightarrow 0. \quad (3.14)$$

Proof. Using local coordinates, it can be proved that the map

$$X^u \frac{\partial}{\partial x^u} + X^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} \xrightarrow{C'} \left(X^u + \frac{\partial C^u}{\partial y^{\bar{u}}} X^{\bar{u}} \right) \frac{\partial}{\partial x^u} \quad (3.15)$$

gives a left splitting of I_0'' . \square

It follows thus an inclusion of $\pi_{N\mathcal{F}}^* N\mathcal{F}$ in $\pi_{N\mathcal{F}}^* TM$, via the injective morphism D'' , that gives a decomposition

$$\pi_{N\mathcal{F}}^* TM = \pi_{N\mathcal{F}}^* T\mathcal{F} \oplus N''\mathcal{F}, \quad (3.16)$$

where $N''\mathcal{F} = D''(\pi_{N\mathcal{F}}^* N\mathcal{F})$.

Let us deal now with an implicit realization of nonlinear constraints.

Let $F : TM \rightarrow T\mathcal{F}$ be a fibered manifold map. Using local coordinates, F has the form

$$(x^u, x^{\bar{u}}, y^u, y^{\bar{u}}) \xrightarrow{F} (x^u, x^{\bar{u}}, F^u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}})).$$

Let us notice that the property of a point $z \in TM$, of coordinates $(x^v, x^{\bar{v}}, y^v, y^{\bar{v}})$, to have $F^u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}}) = 0$, does not depend on coordinates; we say that a such point z is a *constraint point*.

We also say that F is a *contravariant implicit constraint* (or a *con-constraint* for short) if

1. for every $x \in M$ and any transverse vector $\bar{X}_x \in N_x\mathcal{F}$, there is a constraint point in $T_x M$ that projects on \bar{X}_x ;
2. the local matrices $\left(\frac{\partial F^u}{\partial y^v}(z) \right)$ are non-singular in all constraint points z .

Using the implicit mapping theorem, and local coordinates, these conditions read that the local equations $F^u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}}) = 0$ can be solved with respect to y^v , giving local functions $(x^u, x^{\bar{u}}, y^{\bar{u}}) \rightarrow C^u(x^u, x^{\bar{u}}, y^{\bar{u}})$ in a neighborhood of any point in $N\mathcal{F}$, such that $F^u(x^v, x^{\bar{v}}, C^v, y^{\bar{v}}) = 0$. Finally, we obtain local nonlinear constraints $C_U : N\mathcal{F}_U \rightarrow TU$, where $U \subset M$ are open sets that cover M .

Let us consider now the covariant case.

Let $G : TM \rightarrow T^*\mathcal{F}$ be a fibered manifold map. Using local coordinates, G has the form

$$(x^u, x^{\bar{u}}, y^u, y^{\bar{u}}) \xrightarrow{G} (x^u, x^{\bar{u}}, G_u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}})).$$

As in the contravariant case, the property of a point $z \in TM$, called also a *constraint point*, of coordinates $(x^v, x^{\bar{v}}, y^v, y^{\bar{v}})$, to have $G_u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}}) = 0$, does not depend on coordinates.

We say that G is a *covariant implicit constraint* (or a *cov-constraint* for short) if

1. for every $x \in M$ and any transverse vector $\bar{X}_x \in N_x\mathcal{F}$, there is a constraint point in $T_x M$ that projects on \bar{X}_x ;
2. the local matrices $\left(\frac{\partial G_u}{\partial y^v}(z) \right)$ are non-singular in all constraint points z .

These conditions read that the local equations $G_u(x^v, x^{\bar{v}}, y^v, y^{\bar{v}}) = 0$ can be solved with respect to y^u , giving local functions $(x^u, x^{\bar{u}}, y^{\bar{u}}) \rightarrow C^u(x^u, x^{\bar{u}}, y^{\bar{u}})$ in a neighborhood of any point in $N\mathcal{F}$, such that $G_u(x^v, x^{\bar{v}}, C^v, y^{\bar{v}}) = 0$. Finally, as in the contravariant case, we obtain local nonlinear constraints $C_U : N\mathcal{F}_U \rightarrow TU$, where $U \subset M$ are open sets that cover M .

The implicit form of constraints can be used to give an invariant form to the condition that a covector type be a combination of partial derivatives of functions that give the constraints; for example, in nonholonomic mechanics, the Chetaev condition reads that the covector giving the Euler-Lagrange derivative is such a combination.

Proposition 3.2. *Let $\pi_0 : N \rightarrow M$ be a fibered manifold over M and $E : N \rightarrow T^*M$ be a fibered manifold map. If $E = E_u dx^u + E_{\bar{u}} dx^{\bar{u}}$ has the property*

$$E_u = \sum_v \lambda_v \frac{\partial F_v}{\partial y^u}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}), E_{\bar{u}} = \sum_v \lambda_v \frac{\partial F_v}{\partial y^{\bar{u}}}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}),$$

for cov-constraints F , or

$$E_u = \sum_v \lambda^v \frac{\partial F_v}{\partial y^u}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}), E_{\bar{u}} = \sum_v \lambda^v \frac{\partial F_v}{\partial y^{\bar{u}}}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}),$$

for con-constraints F , then one have

$$E_{\bar{u}} + \sum_u \frac{\partial Q^u}{\partial y^{\bar{u}}} E_u = 0.$$

Proof. We consider the con-constraints case, since the cov-constraints case is analogous. Differentiating the implicit equation $F_u(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}) = 0$ with respect to $y^{\bar{u}}$, we obtain

$$\sum_v \frac{\partial Q^v}{\partial y^{\bar{u}}} \frac{\partial F_u}{\partial y^v}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}) + \frac{\partial F_u}{\partial y^{\bar{u}}}(x^u, x^{\bar{u}}, Q^u, y^{\bar{u}}) = 0,$$

thus using the hypothesis, the conclusion follows. \square

Nonlinear constraints lifts to linear constraints of the natural lifted foliation $\mathcal{F}_{N\mathcal{F}}$ on $N\mathcal{F}$, as follows. On an intersection of two adapted charts, the rule is

$$C^{u'}(x^{v'}, x^{\bar{v}'}, y^{\bar{v}'}) = \frac{\partial x^{u'}}{\partial x^u} C^u(x^v, x^{\bar{v}}, y^{\bar{v}}) + \frac{\partial x^{u'}}{\partial x^{\bar{u}}} y^{\bar{u}}. \quad (3.17)$$

Using this formula, by a direct computation, one can check that the formulas $\mathcal{C}_{\bar{u}}^u = \frac{\partial C^u}{\partial y^{\bar{u}}}$, $\mathcal{C}_v^u = 0$ gives rise a linear constraint on $\mathcal{F}_{N\mathcal{F}}$, i.e. a splitting (left \mathcal{C} or right \mathcal{D}) of the exact sequence

$$0 \rightarrow T\mathcal{F}_{N\mathcal{F}} \xrightarrow{I'_0} T(N\mathcal{F}) \xrightarrow{\Pi'_0} N\mathcal{F}_{N\mathcal{F}} \rightarrow 0. \quad (3.18)$$

If $C : TM \rightarrow T\mathcal{F}$ is a linear constraint, it is a nonlinear one as well. Indeed, the right splitting $D : N\mathcal{F} \rightarrow TM$ induce by $\pi_{N\mathcal{F}}^* D : \pi_{N\mathcal{F}}^* N\mathcal{F} \rightarrow \pi_{N\mathcal{F}}^* TM$, the vector field $C' = \pi_{N\mathcal{F}}^* D(\Gamma_0)$ that is a nonlinear constraint.

An affine constraint gives rise also to a nonlinear one, in a similar way. Indeed, an affine constraint is given by a linear constraint C and a vector field $b \in \Gamma(T\mathcal{F})$. The vector field $C' = \pi_{N\mathcal{F}}^* D(\Gamma_0) + \pi_{N\mathcal{F}}^* b$ gives a nonlinear constraint, where D is the right splitting of (3.2) corresponding to C .

We see below that linear and affine constraint have in common a curvature that is a tensor.

An *almost transverse semi-spray* is a (non-necessarily foliated) section $S : N\mathcal{F} \rightarrow NN\mathcal{F}$ that is a section for the both structures of vector bundle of $NN\mathcal{F}$ over $N\mathcal{F}$ (one of usual vector bundle,

the other one induced by the transversal component of the differential of the canonical projection $N\mathcal{F} \rightarrow M$, as a foliated map). In the case of the trivial foliation by the points of M , we recover the definition of a semi-spray on M . Using local coordinates, an almost transverse semi-spray S has the local form

$$(x^u, x^{\bar{u}}, y^{\bar{u}}) \xrightarrow{\bar{S}} (x^u, x^{\bar{u}}, y^{\bar{u}}, S^{\bar{u}}(x^u, x^{\bar{u}}, y^{\bar{u}})). \quad (3.19)$$

If we ask that S be a foliate section, we say that S is a transverse semi-spray. The only difference in formula (3.19) is that $S^{\bar{u}} = S^{\bar{u}}(x^{\bar{u}}, y^{\bar{u}})$.

In order to lift of an (almost) transverse semi-spray one need a nonlinear constraint (in particular it can be a linear or an affine one).

Proposition 3.3. *If $\bar{S} \in \Gamma(N\widetilde{N\mathcal{F}})$ is an almost transverse semi-spray and $C : N\mathcal{F} \rightarrow TM$ is a nonlinear constraint, then there is a unique vector field $S \in \mathcal{X}(\widetilde{N\mathcal{F}})$ that projects by $T\widetilde{N\mathcal{F}} \rightarrow N\widetilde{N\mathcal{F}}$ and $T\widetilde{N\mathcal{F}} \rightarrow TM$ to S and C respectively.*

Proof. We use local coordinates $(x^u, x^{\bar{u}}, y^{\bar{u}})$ on an open set $V = \pi_{N\mathcal{F}}^{-1}U \subset N\mathcal{F}$, corresponding to some coordinates $(x^u, x^{\bar{u}})$ on $U \subset M$. Let \bar{S} and C having the local forms (3.19) and (3.13) respectively. Taking into account the conditions, then S has the local form

$$\begin{aligned} S &= C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}} + S^{\bar{u}} \frac{\partial}{\partial y^{\bar{u}}} \\ &= C_V + S^{\bar{u}} \frac{\partial}{\partial y^{\bar{u}}}. \end{aligned} \quad (3.20)$$

By a straightforward verification of chain rules on the intersection domains, one can check that S is a global vector field. \square

Notice that considering coordinates $(x^u, x^{\bar{u}}, y^{\bar{u}})$ and $(x^{u'}, x^{\bar{u}'}, y^{\bar{u}'})$ on $V = \pi_{N\mathcal{F}}^{-1}U$ and $V' = \pi_{N\mathcal{F}}^{-1}U'$ respectively, then

$$C_V = C_{V'} + y^{\bar{v}} \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{v}}} \frac{\partial}{\partial y^{\bar{u}'}} \quad (3.21)$$

$$S^{\bar{u}'}(x^{v'}, x^{\bar{v}'}, y^{\bar{v}'}) = S^{\bar{u}}(x^v, x^{\bar{v}}, y^{\bar{v}}) \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}} + y^{\bar{v}} \frac{\partial y^{\bar{u}'}}{\partial x^{\bar{v}}}. \quad (3.22)$$

A vector field $S \in \mathcal{X}(N\mathcal{F})$ given by Proposition 3.3 will be called a C -semispray.

Let us notice that C_V and C have the same formulas, but they are different as vector fields; $C : N\mathcal{F} \rightarrow TM$, but $C_V \in \mathcal{X}(V) = \mathcal{X}(N\mathcal{F}_U)$ is a local vector field.

As a representative nonlinear example, we consider the Appell's example of nonlinear constraint. Take the foliation of $\mathbb{R}_0^3 = \mathbb{R}^3 \setminus \{0\}$ generated by $\frac{\partial}{\partial z}$. Denote $x = x^{\bar{1}}$, $y = x^{\bar{2}}$ and $z = x^1$ and consider the nonlinear constraint given by the implicit equation $\alpha^2 \left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right) - (x^1)^2 = 0$, $\alpha \neq 0$.

We have $C^3(y^{\bar{1}}, y^{\bar{2}}) = \pm \alpha \sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}$, but we take $C^3(y^{\bar{1}}, y^{\bar{2}}) = \alpha \sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}$.

Formula (3.15) gives

$$X^1 \frac{\partial}{\partial x^1} + X^{\bar{1}} \frac{\partial}{\partial x^{\bar{1}}} + X^{\bar{2}} \frac{\partial}{\partial x^{\bar{2}}} \xrightarrow{C'} \left(X^1 + \alpha \frac{X^{\bar{1}} y^{\bar{1}} + X^{\bar{2}} y^{\bar{2}}}{\sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}} \right) \frac{\partial}{\partial x^1}.$$

We can consider time dependent constraints, as follows. Let us consider $N^T\mathcal{F} = N\mathcal{F} \times \mathbb{R}$ or $N^T\mathcal{F} = N\mathcal{F} \times S^1$ and the foliation $\overline{\mathcal{F}}^T$ on $N^T\mathcal{F}$ is induced by the foliation $\overline{\mathcal{F}} = \mathcal{F}_{N\mathcal{F}}$ on $N\mathcal{F}$, such that the canonical projection $N^T\mathcal{F} \rightarrow N\mathcal{F}$ is a diffeomorphism of leaves, thus the new parameter is transverse.

A *time dependent nonlinear constraint* on M is a map $C : N^T\mathcal{F} \rightarrow TM$, viewed also as a section $C \in \Gamma(\pi_{N^T\mathcal{F}}^*TM)$, such that $\tilde{J}(C) = \Gamma_0$. Using local coordinates,

$$(x^u, x^{\bar{u}}, y^{\bar{u}}, t) \xrightarrow{C} (C^u(x^v, x^{\bar{v}}, y^{\bar{v}}, t), y^{\bar{u}}), C = C^u \frac{\partial}{\partial x^u} + y^{\bar{u}} \frac{\partial}{\partial x^{\bar{u}}}. \quad (3.23)$$

There is an exact sequence, induced by (3.14):

$$0 \rightarrow \pi_{N^T\mathcal{F}}^*T\mathcal{F} \xrightarrow{I_0''} \pi_{N^T\mathcal{F}}^*TM \xrightarrow{\Pi_0''} \pi_{N^T\mathcal{F}}^*N\mathcal{F} \rightarrow 0. \quad (3.24)$$

As in the time independent case, a time dependent nonlinear constraint on M gives also rise to a left splitting C'' or, equivalently, a right splitting D'' of the exact sequence (3.24); using local coordinates, then analogous formulas (3.15) and (3.16) holds.

A more general approach of time dependent constraints can be considered taking $M' = M \times \mathbb{R}$ instead M and the parameter from \mathbb{R} being transverse. Then transverse coordinates get $x^{\bar{u}}$, where $\bar{u} = \overline{0, n}$ and $x^{\bar{0}} = t \in \mathbb{R}$. The case considered above is when $y^{\bar{0}} = 1$, corresponding to $(t, 1) \equiv \frac{\partial}{\partial t}$, the tangent vector to curve $t \rightarrow t$ in \mathbb{R} . We do not use this general situation in the paper.

A classical example of time dependent nonlinear constraint is the Appell-Hammel dynamic system in an elevator considered in [19], having the time dependent constraints

$$\alpha^2 \left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right) - (x^1 - v^0(t))^2 = 0. \quad (3.25)$$

It is easy to see that the above Appell example corresponds to the particular case when $v^0(t) = 0$.

We have $x^1 = C^1(y^{\bar{1}}, y^{\bar{2}}) = v^0(t) \pm \alpha \sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}$; we take $C^1(y^{\bar{1}}, y^{\bar{2}}) = v^0(t) + \alpha \sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}$.

Formula (3.15) gives

$$X^1 \frac{\partial}{\partial x^1} + X^{\bar{1}} \frac{\partial}{\partial x^{\bar{1}}} + X^{\bar{2}} \frac{\partial}{\partial x^{\bar{2}}} \xrightarrow{C'} \left(X^1 + \alpha \frac{X^{\bar{1}}y^{\bar{1}} + X^{\bar{2}}y^{\bar{2}}}{\sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}} \right) \frac{\partial}{\partial x^1}.$$

Notice that formula (3.17) on $T(N\mathcal{F})$ shows that $\frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}} \frac{\partial x^{\bar{v}'}}{\partial x^{\bar{v}}} \frac{\partial^2 C^{u'}}{\partial y^{\bar{u}'} \partial y^{\bar{v}'}} = \frac{\partial x^{u'}}{\partial x^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}}$, thus

$$\mathcal{C} = \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} dx^{\bar{u}} \otimes dx^{\bar{v}} \otimes \frac{\partial}{\partial x^u}$$

defines a tensor $\mathcal{C} \in L(VN\mathcal{F} \otimes VN\mathcal{F}, TN\mathcal{F})$. This tensor vanishes only for linear or affine constraint.

In both nonlinear Appell's examples, the matrix of \mathcal{C} is

$$\left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right)^{-\frac{3}{2}} \begin{pmatrix} (y^{\bar{2}})^2 & -y^{\bar{1}}y^{\bar{2}} \\ -y^{\bar{1}}y^{\bar{2}} & (y^{\bar{1}})^2 \end{pmatrix}. \quad (3.26)$$

Proposition 3.4. *If $C : N\mathcal{F} \rightarrow TM$ is a nonlinear constraint and $S \in \mathcal{X}(N\mathcal{F})$ is a C -semispray, then the local formula*

$$-\left[\left[\frac{\partial}{\partial y^{\bar{u}}}, C_V\right], C_V\right] + S^{\bar{v}} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{\partial}{\partial x^u} = R_{\bar{u}}^u \frac{\partial}{\partial x^u}, \quad (3.27)$$

gives a global tensor $R \in L(VN\mathcal{F}, T\mathcal{F}_{N\mathcal{F}})$, $R = R_{\bar{u}}^u (x^v, x^{\bar{v}}, y^{\bar{v}}) \frac{\partial}{\partial x^u} \otimes dx^{\bar{u}}$.

Proof. Let us consider two coordinates systems on V and V' , $V \cap V' \neq \emptyset$, on $N\mathcal{F}$, as in the proof of Proposition 3.3. One can check that

$$\left[\frac{\partial}{\partial y^{\bar{u}}}, C_V\right] - \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}} \left[\frac{\partial}{\partial y^{\bar{u}'}} , C_{V'}\right] = \frac{\partial y^{\bar{u}'}}{\partial x^{\bar{u}}} \frac{\partial}{\partial y^{\bar{u}'}}.$$

We have

$$-\left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}}\right]\right] = \left(C^v \frac{\partial^2 C^u}{\partial x^v \partial y^{\bar{u}}} + y^{\bar{v}} \frac{\partial^2 C^u}{\partial x^{\bar{v}} \partial y^{\bar{u}}} - \frac{\partial C^u}{\partial x^{\bar{u}}} - \frac{\partial C^v}{\partial y^{\bar{u}}} \frac{\partial C^u}{\partial x^v}\right) \frac{\partial}{\partial x^u}. \quad (3.28)$$

By a long and straightforward computation, one obtain the formula

$$-\left[\left[\frac{\partial}{\partial y^{\bar{u}}}, C_V\right], C_V\right] + \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}} \left[\left[\frac{\partial}{\partial y^{\bar{u}'}} , C_{V'}'\right], C_{V'}'\right] = y^{\bar{v}} \frac{\partial y^{\bar{v}'}}{\partial x^{\bar{v}}} \frac{\partial^2 C^{v'}}{\partial y^{\bar{v}'} \partial x^{\bar{u}'}} \frac{\partial x^{\bar{u}'}}{\partial x^{\bar{u}}} \frac{\partial}{\partial x^{v'}}.$$

Using also formula (3.21), we obtain the conclusion. \square

We call R given by Proposition 3.4 as the S -curvature of C ; this R is free on S only in the case when $\mathcal{C} = 0$, i.e. when C is a linear or affine constraint, as in [5, 4], (see the formulas (4.34) and (4.35) below). In general, the formula

$$\frac{\partial}{\partial y^{\bar{u}}} \xrightarrow{R_V} -\left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}}\right]\right]$$

gives only a local linear map $L(VN\mathcal{F}_U, T\mathcal{F}_{N\mathcal{F}_U})$ that does not extends to $L(VN\mathcal{F}, T\mathcal{F}_{N\mathcal{F}})$. We say that R_V is the *pseudo-curvature* of C ; it is not a tensor.

In the case of Appell's nonlinear constraint, one have $R_V = 0$, only using the euclidean coordinates.

4 The Lagrangian dynamics for linear, affine and nonlinear constraint systems

In this section we look closer to properties that involve together Lagrangians and linear constraint on foliations, following [5, 4]. Specifically, the dynamics of the system is ruled by a master Lagrangian $L : \widetilde{TM} \rightarrow \mathbb{R}$ and a linear or affine constraint $C : TM \rightarrow T\mathcal{F}$, or $D : N\mathcal{F} \rightarrow TM$, considered in the previous section.

Let $L : TM \rightarrow \mathbb{R}$ be a Lagrangian on the total space of a foliated manifold endowed with a system of a nonlinear (possible linear or affine) constraint. We study the case of nonlinear constraints, thus we consider one given by a left splitting \mathcal{C} of I'_0 , or by a right splitting \mathcal{D} of the projection Π'_0 in the exact sequence (3.18). As in the case of linear or affine constraints

in [4, Sect. 5.2], we consider that the equations of motions governed by a Lagrangian and the constraint, can be deduced imposing the principle to apply first the variation, then the projection of the Lagrange equations according to the constraint, adapting d'Alembert principle. Specifically, using the decomposition (3.16), then the constraints effect on Lagrange equations has, as for linear and affine constraints in [4, Sect. 5.2], the forms

$$\begin{aligned} \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) \delta x^u + \left(\frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} \right) \delta x^{\bar{u}} &= 0, \\ \delta x^u + C_{\bar{u}}^u \delta x^{\bar{u}} &= 0, \end{aligned}$$

where $C_{\bar{u}}^u = \frac{\partial C^u}{\partial y^{\bar{u}}}$. Notice that δ is subject to $t = \text{const.}$ Substituting $\delta x^{\bar{u}}$ in the Lagrange equations, one obtain the induced *constrained Lagrange equations*:

$$\left(\frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} \right) - C_{\bar{u}}^u \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) = 0. \quad (4.29)$$

As we see below for implicit nonlinear constraints, these equations are concordant to Chetaev conditions.

A nonlinear constraint $C \in \Gamma(\pi_{N\mathcal{F}}^* TM)$ can be viewed as a map $C : N\mathcal{F} \rightarrow TM$, thus any Lagrangian $L : TM \rightarrow \mathbb{R}$ induces by composition $N\mathcal{F} \xrightarrow{C} TM \xrightarrow{L} \mathbb{R}$ a new Lagrangian $L_c = L \circ C$ on $N\mathcal{F}$, called the *constrained Lagrangian*:

$$L_c(x^u, x^{\bar{u}}, y^{\bar{u}}) = L(x^u, x^{\bar{u}}, C^u, y^{\bar{u}}). \quad (4.30)$$

According to [4, Sect. 5.2], in the cases when

$$\begin{aligned} C^u(x^v, x^{\bar{v}}, y^{\bar{v}}) &= y^{\bar{u}} C_{\bar{u}}^u(x^v, x^{\bar{v}}), \\ C^u(x^v, x^{\bar{v}}, y^{\bar{v}}) &= y^{\bar{u}} C_{\bar{u}}^u(x^v, x^{\bar{v}}) + b^u(x^v, x^{\bar{v}}) \end{aligned}$$

i.e. of linear and affine constraints respectively, the constrained Lagrange equations (4.29) can be written in terms of the constrained Lagrangian as

$$\left(\frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} \right) = C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} - \frac{\partial L}{\partial y^u} (B_{\bar{u}\bar{v}}^u y^{\bar{v}} + \gamma_{\bar{u}}^u), \quad (4.31)$$

where

$$B_{\bar{u}\bar{v}}^u = C_{\bar{u}}^v \frac{\partial C_{\bar{v}}^u}{\partial x^v} - C_{\bar{v}}^v \frac{\partial C_{\bar{u}}^u}{\partial x^v} + \frac{\partial C_{\bar{v}}^u}{\partial x^{\bar{u}}} - \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}}, \quad (4.32)$$

$$\gamma_{\bar{u}}^u = \frac{\partial b^u}{\partial x^{\bar{u}}} + C_{\bar{u}}^v \frac{\partial b^u}{\partial x^v} - b^v \frac{\partial C_{\bar{u}}^u}{\partial x^v} \quad (4.33)$$

are both tensors, B and γ (see [5, 4] for more details).

In the linear constraint case (i.e. $b^u = 0$), the formula (4.32) gives, according to formula (3.28), that the curvature R of C is

$$R_{\bar{u}}^u = y^{\bar{v}} B_{\bar{v}\bar{u}}^u. \quad (4.34)$$

In the affine constraint case, the formulas (4.32), (4.33) and (3.28) give the curvature of C by

$$R_{\bar{u}}^u = y^{\bar{v}} B_{\bar{v}\bar{u}}^u + \gamma_{\bar{u}}^u. \quad (4.35)$$

In the sequel we extend formulas (4.31) to the case of nonlinear constraints.
The equation of motion of the extended nonholonomic system is

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} + C_{\bar{u}}^u \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) \right) dx^{\bar{u}} = 0.$$

In the case of the induced Lagrangian L_c , one have

$$-\delta L_c = \left(\frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} \right) dx^{\bar{u}}.$$

We have, for $C_{\bar{u}}^u = \frac{\partial C^u}{\partial y^{\bar{u}}}$,

$$\begin{aligned} \frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} &= \frac{d}{dt} \left(\frac{\partial L}{\partial y^{\bar{u}}} + C_{\bar{u}}^u \frac{\partial L}{\partial y^u} \right) = \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} + \frac{\partial L}{\partial y^u} \frac{d}{dt} C_{\bar{u}}^u + C_{\bar{u}}^u \frac{d}{dt} \frac{\partial L}{\partial y^u}, \\ \frac{\partial L_c}{\partial x^{\bar{u}}} &= \frac{\partial L}{\partial x^{\bar{u}}} + \frac{\partial C^v}{\partial x^{\bar{u}}} \frac{\partial L}{\partial y^v}, \quad \frac{\partial L_c}{\partial x^u} = \frac{\partial L}{\partial x^u} + \frac{\partial C^v}{\partial x^u} \frac{\partial L}{\partial y^v}, \end{aligned}$$

thus

$$\begin{aligned} -\delta L_c &= \frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} \\ &= \frac{d}{dt} \frac{\partial L}{\partial y^{\bar{u}}} - \frac{\partial L}{\partial x^{\bar{u}}} + C_{\bar{u}}^u \left(\frac{d}{dt} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial x^u} \right) + \frac{\partial L}{\partial y^u} \left(\frac{d}{dt} C_{\bar{u}}^u - \frac{\partial C^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial C^u}{\partial x^v} \right) \\ &= -\delta L + \frac{\partial L}{\partial y^u} \left(\frac{\partial C_{\bar{u}}^u}{\partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} + \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}} y^{\bar{v}} + \frac{\partial C_{\bar{u}}^u}{\partial x^v} C^v - \frac{\partial C^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial C^u}{\partial x^v} \right) \\ &= \frac{\partial L}{\partial y^u} \left(\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} + \frac{\partial C_{\bar{u}}^u}{\partial x^{\bar{v}}} y^{\bar{v}} + \frac{\partial C_{\bar{u}}^u}{\partial x^v} C^v - \frac{\partial C^u}{\partial x^{\bar{u}}} - C_{\bar{u}}^v \frac{\partial C^u}{\partial x^v} \right). \end{aligned}$$

Thus, using (3.28), one have

$$-\delta L_c = \frac{\partial L}{\partial y^u} \left(\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} - \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u \right) dx^{\bar{u}}. \quad (4.36)$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} &= \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} + \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial x^{\bar{v}}} y^{\bar{v}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} \\ &= \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} + F_{\bar{u}}, \end{aligned}$$

thus

$$-\delta L_c = \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} + F_{\bar{u}}, \quad (4.37)$$

where

$$F_{\bar{u}} = \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial x^{\bar{v}}} y^{\bar{v}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u}. \quad (4.38)$$

Comparing the relations (4.36) and (4.37), we obtain

$$\left(\frac{\partial L}{\partial y^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \right) \frac{dy^{\bar{v}}}{dt} - F_{\bar{u}} - \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u = 0. \quad (4.39)$$

Denote

$$h_{\bar{u}\bar{v}} = \frac{\partial L}{\partial y^u} \frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}}. \quad (4.40)$$

It is easy to see that $h = (h_{\bar{u}\bar{v}})$ gives a global bilinear form in the fibers of $VN\mathcal{F} = \pi_{N\mathcal{F}}^* N\mathcal{F}$.

We have, by a straightforward computation,

$$h_{\bar{u}\bar{v}} = \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^{\bar{v}}} + \frac{\partial C^u}{\partial y^{\bar{u}}} \frac{\partial^2 L}{\partial y^{\bar{v}} \partial y^u} + \frac{\partial C^u}{\partial y^{\bar{v}}} \frac{\partial^2 L}{\partial y^{\bar{u}} \partial y^u} + \frac{\partial C^u}{\partial y^{\bar{u}}} \frac{\partial C^v}{\partial y^{\bar{v}}} \frac{\partial^2 L}{\partial y^u \partial y^v}.$$

Using the splitting (C'' at left or D'' at right) of exact the sequence (3.14) given by Proposition 3.1, one can easy deduce an interpretation of h . Recall that the Hessian of L is a bilinear form in the fibers of $VTM = \pi_{TM}^* TM$.

Proposition 4.1. *The bilinear form h has the form $h = (D'')^* H_L$, where H_L is the vertical hessian of L , restricted to $N\mathcal{F}$, as the image of the constraint map $C : N\mathcal{F} \rightarrow TM$.*

Proof. We use adapted coordinates. The conclusion follows using the form (3.15) and the identity

$$(h_{\bar{u}\bar{v}}) = \begin{pmatrix} \delta_{\bar{u}}^{\bar{u}_1} & C_{\bar{u}}^{u_1} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 L}{\partial y^{\bar{u}_1} \partial y^{\bar{u}_2}} & \frac{\partial^2 L}{\partial y^{\bar{u}_1} \partial y^{u_2}} \\ \frac{\partial^2 L}{\partial y^{u_1} \partial y^{\bar{u}_2}} & \frac{\partial^2 L}{\partial y^{u_1} \partial y^{u_2}} \end{pmatrix} \begin{pmatrix} \delta_{\bar{v}}^{\bar{u}_2} \\ C_{\bar{v}}^{u_2} \end{pmatrix}.$$

□

We say that the Lagrangian L is C -regular if the bilinear form h is nondegenerted on the fibers of $VN\mathcal{F}$. If it is the case, denoting

$$(h^{\bar{u}\bar{v}}) = (h_{\bar{u}\bar{v}})^{-1},$$

the equation (4.39) gives

$$\begin{aligned} \frac{dy^{\bar{u}}}{dt} &= S^{\bar{u}} \stackrel{def.}{=} h^{\bar{u}\bar{v}} \left(F_{\bar{v}} + \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u \right) \\ &= h^{\bar{u}\bar{v}} \left(\frac{\partial^2 L_c}{\partial y^{\bar{v}} \partial x^{\bar{w}}} y^{\bar{w}} - \frac{\partial L_c}{\partial x^{\bar{v}}} - C_{\bar{v}}^u \frac{\partial L_c}{\partial x^u} + \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{v}}} \right] \right]^u \right). \end{aligned} \quad (4.41)$$

By a straightforward computation, based on the equality (4.39), one can prove that the local functions $(S^{\bar{u}})$ verify the rule (3.22) on the intersection of compatible domains, giving by formula (3.19) an almost transverse semi-spray S , called *canonically associated* with C and L . Using Proposition 3.3 and the above constructions, one have the following result.

Proposition 4.2. *If the Lagrangian L is C -regular, then the integral curves of the equation of motion of the extended nonholonomic system are the integral curves of a C -semispray \mathcal{S} .*

Notice that in the particular case of linear and affine constraints, using formulas (3.28), then (4.31) can be deduced from (4.41).

The Legendre map of L , $\mathcal{L} : TM \rightarrow T^*M$, or $\mathcal{L} \in \Gamma(\pi_{TM}^* T^*M)$, is given in local coordinates by

$$\mathcal{L} = \frac{\partial L}{\partial y^u} dy^u + \frac{\partial L}{\partial y^{\bar{u}}} dy^{\bar{u}}.$$

The statement below gives the equations of motion in the same form the C -regular case and the linear and affine cases of constraints, studied in [5, 4].

Theorem 4.1. *Let $L : TM \rightarrow \mathbb{R}$ be a Lagrangian, $C : NF \rightarrow TM$ be a nonlinear constraint and $L_c = L \circ C$ be the constrained Lagrangian. If one of the following conditions holds:*

1. L is C -regular, or
2. C is a linear constraint, or
3. C is an affine constraint

then the constrained Lagrange equations (4.29) have the form

$$\delta L_c = \langle R, \mathcal{L} \rangle,$$

or, using local coordinates,

$$\frac{d}{dt} \frac{\partial L_c}{\partial y^{\bar{u}}} - \frac{\partial L_c}{\partial x^{\bar{u}}} + C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u} = \frac{\partial L}{\partial y^u} R_{\bar{u}}^u,$$

where R is the S -curvature of C and S is the almost transverse semi-spray S canonically associated, in the first case, or R is the curvature of C , in the last two cases.

In the case of time dependent constraints (as in [19], see the Example 3 in the next section), but a time independent lagrangian, the equations of motion are obtained in the same way as equations (4.36), taking into account the fact that the constraints are time dependent, but the lagrangian is not. One obtain that equations (4.36) are replaced by

$$-\delta L_c = \left(\frac{\partial^2 C^u}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \frac{dy^{\bar{v}}}{dt} \frac{\partial L}{\partial y^u} + \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u} - \frac{\partial L}{\partial y^u} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^u \right) dx^{\bar{u}}.$$

and the equations (4.39) are valid in the same form, but with

$$F_{\bar{u}} = \frac{\partial^2 C^u}{\partial t \partial y^{\bar{u}}} \frac{\partial L}{\partial y^u} - \frac{\partial^2 L_c}{\partial t \partial y^{\bar{u}}} + \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial x^{\bar{v}}} y^{\bar{v}} - \frac{\partial L_c}{\partial x^{\bar{u}}} - C_{\bar{u}}^u \frac{\partial L_c}{\partial x^u}. \quad (4.42)$$

5 Examples

Example 1. We consider first the case of Appell's linear constraints.

Let us consider the lagrangian

$$L(x^1, x^{\bar{1}}, x^{\bar{2}}, y^1, y^{\bar{1}}, y^{\bar{2}}) = \frac{1}{2} \alpha \left((y^1)^2 + (y^{\bar{2}})^2 \right) + \frac{1}{2} \beta (y^3)^2 + \frac{1}{2} I_1 (y^{\bar{1}})^2 + \frac{1}{2} I_2 (y^{\bar{2}})^2 + \gamma x^3.$$

Using the form (3.8) of constraints, the induced lagrangian has the form

$$\begin{aligned} L_c(x^1, x^2, x^3, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) &= \frac{1}{2} (I_1 + \alpha R^2 + \beta r^2) (y^{\bar{1}})^2 + \frac{1}{2} I_2 (y^{\bar{2}})^2 + \gamma x^3 \\ &= \frac{1}{2} \alpha'' (y^{\bar{1}})^2 + \frac{1}{2} I_2 (y^{\bar{2}})^2 + \gamma x^3. \end{aligned}$$

Using formulas (3.9) and (4.34), we have

$$\begin{aligned} R_1^1 &= B_{12}^1 y^2 = -R y^2 \sin x^2, & R_2^1 &= B_{21}^1 y^1 = R y^1 \sin x^2, \\ R_1^2 &= B_{12}^2 y^2 = R y^2 \cos x^2, & R_2^2 &= B_{21}^2 y^1 = -R y^1 \cos x^2, \\ R_1^3 &= B_{21}^3 y^2 = 0, & R_2^3 &= B_{12}^3 y^1 = 0. \end{aligned}$$

In this case $(h_{\bar{u}\bar{v}})$ given by formula (3.26) is the hessian of L_c . Formula (4.41) gives $S^{\bar{u}} = \frac{dy^{\bar{u}}}{dt} = -h^{\bar{u}\bar{v}} C_v^3 \frac{\partial L_c}{\partial x^3}$; by a straightforward computation, one obtain

$$\frac{dy^{\bar{1}}}{dt} = -\alpha'' r, \quad \frac{dy^{\bar{2}}}{dt} = 0.$$

Since $y^{\bar{u}} = \frac{dy^{\bar{u}}}{dt}$, we obtain $x^{\bar{1}} = -\frac{r\alpha'' t^2}{2} + y_0^{\bar{1}} t + x_0^{\bar{1}}$, $x^{\bar{2}} = y_0^{\bar{2}} t + x_0^{\bar{2}}$; using constraint equations (3.8), one obtain $x^1 = -R(r\alpha'' t + y_0^{\bar{1}}) \cos(y_0^{\bar{2}} t + x_0^{\bar{2}})$, $x^2 = -R(r\alpha'' t + y_0^{\bar{1}}) \sin(y_0^{\bar{2}} t + x_0^{\bar{2}})$, $C^3 = -r(r\alpha'' t + y_0^{\bar{1}})$.

Example 2. We consider the case of Appell's nonlinear constraints, with lagrangian as, for example, in [17, 12]:

$$L(x^1, x^{\bar{1}}, x^{\bar{2}}, y^1, y^{\bar{1}}, y^{\bar{2}}) = \frac{\beta}{2} \left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right) + \frac{\gamma}{2} (y^1)^2 + \delta x^1. \quad (5.43)$$

The induced lagrangian has the form

$$L_c(x^1, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) = \frac{\beta + \alpha\gamma}{2} \left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right) + \delta x^1,$$

thus formula (4.41) gives $S^{\bar{u}} = \frac{dy^{\bar{u}}}{dt} = -h^{\bar{u}\bar{v}} C_v^1 \frac{\partial L_c}{\partial x^1}$, where $(h^{\bar{u}\bar{v}}) = (h_{\bar{u}\bar{v}})^{-1}$ and $(h_{\bar{u}\bar{v}})$ is given by formula (3.26). By a straightforward computation, one obtain

$$\frac{dy^{\bar{u}}}{dt} = \frac{\alpha' y^{\bar{u}}}{\sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}},$$

where $\alpha' = -\frac{\alpha}{2(\gamma\alpha^2 + \beta)}$. Using polar coordinates $y^{\bar{1}} = \rho \cos \varphi$, $y^{\bar{2}} = \rho \sin \varphi$, it follows that $\frac{d\rho}{dt} = \alpha$, $\frac{d\varphi}{dt} = 0$, thus $\rho = \alpha' t + \rho_0$, $\varphi = \varphi_0$. Since $y^{\bar{u}} = \frac{dx^{\bar{u}}}{dt}$, one have $x^{\bar{1}} = \left(\frac{\alpha' t^2}{2} + \rho_0 t \right) \cos \varphi_0 + x_0^{\bar{1}}$, $x^{\bar{2}} = \left(\frac{\alpha' t^2}{2} + \rho_0 t \right) \sin \varphi_0 + x_0^{\bar{2}}$, $x^1 = \pm \alpha \left(\frac{\alpha' t^2}{2} + \rho_0 t \right) + x_0^1$. The solutions of the constrained Lagrange equations are straight lines; but this is not physically correct in the case of Appell machine (see [21]).

Example 3. In the Appell-Hammel dynamic system in an elevator, considered in 3.25, one correspond the lagrangian (5.43) as in Example 2; thus Example 2 is a particular case of this example, when $v^0(t) = 0$. Using Proposition 3.2, we can infer at this stage that, concerning the

solution of the equation of motion, one obtain the same result as in [19, Section 3.2]. Indeed, we have

$$L_c = \frac{\beta + \alpha^2 \gamma}{2} \left((y^{\bar{1}})^2 + (y^{\bar{2}})^2 \right) + \gamma v^{(0)} \sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2} + \delta x^1 + \frac{1}{2} (v^{(0)})^2,$$

then we have that the pseudo-curvature $R_V = \frac{\partial L}{\partial y^{\bar{u}}} \left[C_V, \left[C_V, \frac{\partial}{\partial y^{\bar{u}}} \right] \right]^{\bar{u}} = 0$ and

$$\begin{aligned} (F_{\bar{u}}) &= \left(-\frac{(\alpha\delta + \gamma\dot{v}^{(1)}) y^{\bar{v}}}{\sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}} \right), \\ \left(\frac{\partial L}{\partial y^{\bar{1}}} \frac{\partial^2 C^1}{\partial y^{\bar{u}} \partial y^{\bar{v}}} - \frac{\partial^2 L_c}{\partial y^{\bar{u}} \partial y^{\bar{v}}} \right)^{-1} (F_{\bar{u}})^t &= \left(\frac{\alpha (\alpha\delta + \gamma\dot{v}^{(1)}) y^{\bar{v}}}{\alpha^2 \gamma + \beta} \right), \end{aligned}$$

thus formula (4.41) gives

$$\ddot{x}^{\bar{u}} = \dot{y}^{\bar{u}} = \frac{(\alpha\delta + \gamma\dot{v}^{(1)}) y^{\bar{u}}}{\sqrt{(y^{\bar{1}})^2 + (y^{\bar{2}})^2}},$$

then

$$\frac{d}{dt} \sqrt{(\dot{x}^{\bar{1}})^2 + (\dot{x}^{\bar{2}})^2} = \frac{\dot{x}^{\bar{1}} \dot{x}^{\bar{1}} + \dot{x}^{\bar{2}} \dot{x}^{\bar{2}}}{\sqrt{(\dot{x}^{\bar{1}})^2 + (\dot{x}^{\bar{2}})^2}} = (\alpha\delta + \gamma\dot{v}^{(1)}) = \frac{d}{dt} (\alpha\delta + \gamma v^{(1)})$$

and one obtain the solution as in [19, Section 3.2].

Example 4. The Benenti mechanism [3] (see also [13]) fit in the following example. Consider the foliation of \mathbb{R}^4 with coordinates (x_1, x_2, y_1, y_2) generated by $\frac{\partial}{\partial x_1}$. Denote $x_2 = x^{\bar{1}}$, $y_1 = x^{\bar{2}}$, $y_2 = x^{\bar{3}}$ and $x_1 = x^1$ and consider the nonlinear constraint given by the implicit equation $y^1 y^{\bar{3}} - y^{\bar{1}} y^{\bar{2}} = 0$, $(y^1)^2 + (y^{\bar{1}})^2 + (y^{\bar{2}})^2 + (y^{\bar{3}})^2 \neq 0$, where $(x^1, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}, y^1, y^{\bar{1}}, y^{\bar{2}}, y^{\bar{3}})$ are coordinates on $T\mathbb{R}^4$. We have $C^1(x^1, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}, y^{\bar{1}}, y^{\bar{2}}, y^{\bar{3}}) = \frac{y^{\bar{1}} y^{\bar{2}}}{y^{\bar{3}}}$. Formula (3.15) gives

$$X^1 \frac{\partial}{\partial x^1} + X^{\bar{1}} \frac{\partial}{\partial x^{\bar{1}}} + X^{\bar{2}} \frac{\partial}{\partial x^{\bar{2}}} + X^{\bar{3}} \frac{\partial}{\partial x^{\bar{3}}} \xrightarrow{C'} \left(X^1 + \frac{X^{\bar{1}} y^{\bar{2}} y^{\bar{3}} + X^{\bar{2}} y^{\bar{1}} y^{\bar{3}} - X^{\bar{3}} y^{\bar{1}} y^{\bar{2}}}{(y^{\bar{3}})^2} \right) \frac{\partial}{\partial x^1}.$$

One have $R_V = 0$, only using euclidean coordinates.

One consider the lagrangian

$$L(x^1, x^{\bar{1}}, x^{\bar{2}}, y^1, y^{\bar{1}}, y^{\bar{2}}) = \frac{\alpha}{2} \left((y^1)^2 + (y^{\bar{1}})^2 \right) + \frac{\beta}{2} \left((y^{\bar{2}})^2 + (y^{\bar{3}})^2 \right) + f(x^1, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}),$$

that has the kinetic energy as the original [3], or as in [13, 12]. The induced lagrangian has the form

$$L_c(x^1, x^{\bar{1}}, x^{\bar{2}}, y^{\bar{1}}, y^{\bar{2}}) = \frac{\alpha (y^{\bar{1}})^2}{2 (y^{\bar{3}})^2} \left((y^{\bar{2}})^2 + (y^{\bar{3}})^2 \right) + \frac{\beta}{2} \left((y^{\bar{2}})^2 + (y^{\bar{3}})^2 \right) + f(x^1, x^{\bar{1}}, x^{\bar{2}}, x^{\bar{3}}).$$

Formula (4.38) gives $F_{\bar{1}} = -f_{,\bar{1}} + f_{,1} \frac{y^2 y^3}{(y^1)^2}$, $F_{\bar{2}} = -f_{,\bar{2}} - f_{,1} \frac{y^3}{y^1}$, $F_{\bar{3}} = -f_{,\bar{3}} - f_{,1} \frac{y^2}{y^1}$, where $f_{,1} = \frac{\partial f}{\partial x^1}$ and $f_{,\bar{u}} = \frac{\partial f}{\partial x^{\bar{u}}}$. Then

$$(h^{\bar{u}\bar{v}}) = \begin{pmatrix} -\frac{(y^1)^2((y^1)^2\alpha\beta + (y^1)^2\beta^2 + (y^2)^2\alpha^2 + (y^3)^2\alpha\beta + (y^3)^2\alpha\beta)}{(y^2 y^3)^2\alpha\beta(\alpha+\beta)} & -\frac{y^1}{y^2(\alpha+\beta)} & -\frac{y^1}{y^3\beta} \\ -\frac{y^1}{y^2(\alpha+\beta)} & -\frac{1}{\alpha+\beta} & 0 \\ -\frac{y^1}{y^3\beta} & 0 & -\frac{1}{\beta} \end{pmatrix}$$

and formula (4.41) gives

$$S^{\bar{u}} = \frac{dy^{\bar{u}}}{dt} = h^{\bar{u}\bar{v}} F_{\bar{v}}.$$

In the original case of [3], when the potential f vanish, one obtain $S^{\bar{u}} = 0$, thus the integral curves are straight lines.

Example 5. The following example is the Marle servomechanism [20] (see also [13]), where the Chetaev principle is claimed to fail in the real world. Consider the foliation of \mathbb{R}^2 with coordinates $(x^1, x^{\bar{1}})$ generated by $\frac{\partial}{\partial x^1}$ and consider the nonlinear constraint given by

$$y^1 = f(x^1, x^{\bar{1}}, y^{\bar{1}}).$$

We have $C^3(x^1, x^{\bar{1}}, y^{\bar{1}}) = f(x^1, x^{\bar{1}}, y^{\bar{1}})$. Formula (3.15) gives

$$X^1 \frac{\partial}{\partial x^1} + X^{\bar{1}} \frac{\partial}{\partial x^{\bar{1}}} \xrightarrow{C'} \left(X^1 + X^{\bar{1}} \frac{\partial f}{\partial y^{\bar{1}}} \right) \frac{\partial}{\partial x^1}.$$

One have $R_V = \left[\frac{\partial}{\partial y^{\bar{1}}}, \left[\frac{\partial}{\partial y^{\bar{1}}}, f \frac{\partial}{\partial x^1} + y^{\bar{1}} \frac{\partial}{\partial x^{\bar{1}}} \right] \right] = \frac{\partial^2 f}{\partial (y^{\bar{1}})^2} \frac{\partial}{\partial x^1}.$

One consider the lagrangian

$$L(x^1, x^{\bar{1}}, y^1, y^{\bar{1}}) = \frac{m}{2} \left((y^1)^2 - 2ly^1 y^{\bar{1}} \sin x^{\bar{1}} \right) + \frac{J}{2} (y^{\bar{1}})^2 - mgl \sin x^{\bar{1}},$$

thus

$$L_c(x^1, x^{\bar{1}}, y^{\bar{1}}) = \frac{m}{2} \left(f^2 - 2lf y^{\bar{1}} \sin x^{\bar{1}} \right) + \frac{J}{2} (y^{\bar{1}})^2 - mgl \sin x^{\bar{1}},$$

If one consider on $V = \mathbb{R}^2 \setminus \{(x^1, 0) | x^1 \geq 0\}$ the induced foliation (thus generated by $\frac{\partial}{\partial x^1}$), this is not a locally trivial one and the space of leaves is not Hausdorff separated, thus the use of a foliation is justified in this case.

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